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# Inclusion of short-range order in a mean field theory for the disordered magnetic lattice gas

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**Abstract.** The disordered magnetic lattice gas (DMLG) as a unifying description of many simpler random spin systems has been investigated in an attempt to devise a mean field theory which goes beyond the infinitely-long-ranged model by incorporating short-range order (SRO). We have shown rigorously that the local thermodynamic properties of the DMLG on a Cayley tree of finite coordination number  $z$  are identical to the thermodynamic properties of the DMLG in a pair approximation obtained by using the method of the distribution function. Further, a modified pair approximation for the DMLG is presented which is exactly solvable. It is formulated for general random bond distribution functions, and is then examined for the special case of Gaussian distributions.

## 1. Introduction

Mean field theory, which approximates the behaviour of a system by ignoring the effects of fluctuations, usually provides a good starting point for studying phase transitions in non-random Ising spin systems. Developing useful mean field theories for random spin systems, however, has still remained something of a challenge since the seminal work by Sherrington and Kirkpatrick (SK) [1] who treated their model by means of the  $n$ -replica method. Although the  $n$ -replica method gives exact results at high temperatures [2], it breaks down at low temperatures where it yields a negative entropy. It is nowadays believed that at low temperatures the SK model has a rich ultrametric structure of ‘Gibbs states’ described by Parisi’s replica symmetry breaking solution [3, 4]. Among the various alternative approaches to the SK problem we just mention the TAP equations [5].

Scepticism soon arose about the relevance of a mean field theory based upon an infinite-range model to short-range spin glasses. Consequently, several attempts have appeared in the literature to develop a mean field theory which incorporates some kind of short-range ordering (SRO), i.e. one which takes into account not only the average spin on each site, but also the fluctuations from this average.

A first natural approach was to consider a generalization of the Bethe–Peierls–Weiss (BPW) approximation to random spin systems, or equivalently, to treat the system on a Cayley tree of finite coordination number  $z$ . Since the number of sites on the surface of the Cayley tree is of the same order of magnitude as the total number of sites, surface effects cannot be neglected in the thermodynamic limit. Therefore, as in the non-random case, the BPW approximation and the system on a Cayley tree are only equivalent if one considers suitably defined ‘local’ thermodynamic quantities

rather than the corresponding global or bulk expressions [6–8]. Unfortunately, in the case of random spin systems, the resulting equations for the effective fields constitute an intractable set of coupled integral equations which can only be solved in the limit of infinite coordination number  $z \rightarrow \infty$  (which simply recovers zeroth-order mean field theory) [6, 9–11] or at  $T = 0$  [12, 13].

The technique of the distribution function constitutes an alternative approach to the problem of developing a mean field theory for random spin systems which incorporates SRO. Although different authors adopted it initially, this approach was mainly pursued by groups in Tohoku and later Melbourne. In [14] a comprehensive review of this method in studies of disordered systems up to 1981 is given. After having introduced the disordered magnetic lattice gas (DMLG) as a unifying description of many simpler models for amorphous solids and real fluids, Inawashiro, Frankel and Thompson (IFT) [14, 15] studied this system using a generalized Kikuchi pair approximation [16]. Later, Bell, Frankel and Inawashiro [17] extended this investigation to a general model of a random binary alloy. All these authors formulate the problem self-consistently for finite coordination number  $z$  and derive a set of coupled integral equations for the distribution function of the effective fields and potentials which are analytically rather intractable. They continue their derivation of the order parameters in the limit  $z \rightarrow \infty$ . Thompson, Inawashiro and Frankel (TIF) [11] have shown that in the limit  $z \rightarrow \infty$  the pair approximation for the DMLG is equivalent to the DMLG on a Cayley tree of infinite coordination number if one considers ‘local’ thermodynamic quantities. IFT have further shown that in the limit  $z \rightarrow \infty$  the pair approximation reproduces exactly the SK equations upon a corresponding specialization of the DMLG with Gaussian random bonds.

In this paper we show that the pair approximation for the DMLG, and thus for a whole range of models to which it reduces upon specialization [14, 15], is equivalent to the DMLG on a Cayley tree with ‘local’ interpretation, for arbitrary coordination number  $z$ , i.e. in particular for finite  $z$ . This is done in section 3. In section 4, we then present a modified pair approximation in which the pair cluster is not embedded in self-consistent effective fields and potentials but in which the general shape of the distribution function for these effective fields and potentials is determined by zeroth-order mean field theory (the infinite-range model), while the specifying parameters of this distribution function are determined self-consistently. The advantage of this modified pair approximation is that it can be solved analytically. We formulate the method for general random bond/potential distributions and examine the special case of Gaussian distributions. In section 2, which is orientational, we define our model, introduce our notations and briefly summarize the relevant equations from the IFT pair approximation [14, 15]. Section 5 is devoted to a summary and discussion.

## 2. Model and the pair approximation

The model Hamiltonian for the DMLG is

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij} S_i S_j - \sum_{\langle i,j \rangle} U_{ij} S_i^2 S_j^2 - B \sum_i S_i - \mu \sum_i S_i^2 \quad (2.1)$$

where  $J_{ij}$  denotes the magnetic interaction between lattice sites  $i$  and  $j$ ,  $U_{ij}$  is the potential interaction between lattice sites  $i$  and  $j$ ,  $B$  is an external magnetic field and

$\mu$  is the chemical potential. Spin-1 Ising operators are used to represent a particle with up or down spin at a lattice site by  $S = 1$  or  $S = -1$ , respectively, and a vacant site by  $S = 0$ . It is assumed that  $J_{ij}$  and  $U_{ij}$  are distributed randomly over the lattice bonds with a probability distribution  $\mathcal{P}(J, U)$ .  $\langle i, j \rangle$  runs over all nearest neighbour lattice sites.

The single-site Hamiltonian is introduced as

$$\mathcal{H}_1 = - \left( h_1^{(z)} + B \right) S_1 - \left( l_1^{(z)} + \mu \right) S_1^2 \tag{2.2}$$

where  $h_1^{(z)}$  ( $l_1^{(z)}$ ) represents an effective field (potential) at the site 1 coming from  $z$  neighbours through each interaction bond (see figure 1).

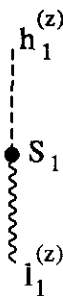


Figure 1. Spin  $S_1$  with the  $z$ -bond field and  $z$ -bond potential which are denoted by a dashed and by a wavy line, respectively.

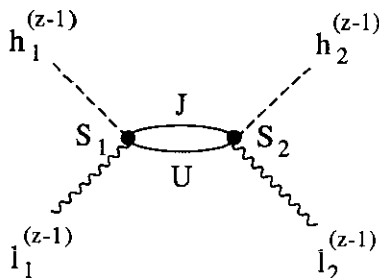


Figure 2. Pair of spins  $S_1$  and  $S_2$  and the corresponding sets of  $(z - 1)$ -bond fields and potentials.

The pair-site Hamiltonian is introduced as

$$\begin{aligned} \mathcal{H}_{12} = & -JS_1S_2 - US_1^2S_2^2 - \left( B + h_1^{(z-1)} \right) S_1 - \left( B + h_2^{(z-1)} \right) S_2 \\ & - \left( \mu + l_1^{(z-1)} \right) S_1^2 - \left( \mu + l_2^{(z-1)} \right) S_2^2 \end{aligned} \tag{2.3}$$

where  $h_i^{(z-1)}$  ( $l_i^{(z-1)}$ ) denotes an effective field (potential) at the  $i$ th site resulting from the  $(z - 1)$  neighbours outside the pair sites (see figure 2).

This allows us to define a single-site density matrix

$$\rho_1 = \exp(-\beta\mathcal{H}_1) \tag{2.4}$$

and a pair-site density matrix as

$$\rho_{12} = \exp(-\beta\mathcal{H}_{12}). \tag{2.5}$$

Evaluating the traces gives

$$\text{Tr}\rho_1 = Z_1 \left( h_1^{(z)}, l_1^{(z)} \right) \tag{2.6}$$

$$\text{Tr}_2\rho_{12} = Z_1 \left( h_2^{(z-1)}, l_2^{(z-1)} \right) \exp \beta \left[ \left( B + h_1^{(z-1)} + H \right) S_1 + \left( \mu + l_1^{(z-1)} + L \right) S_1^2 \right] \tag{2.7}$$

$$\text{Tr}\rho_{12} = Z_1 \left( h_2^{(z-1)}, l_2^{(z-1)} \right) Z_1 \left( h_1^{(z-1)} + H, l_1^{(z-1)} + L \right) \tag{2.8}$$

with

$$Z_1(x, y) = 1 + 2 \exp[\beta(\mu + y)] \cosh \beta(B + x) \tag{2.9}$$

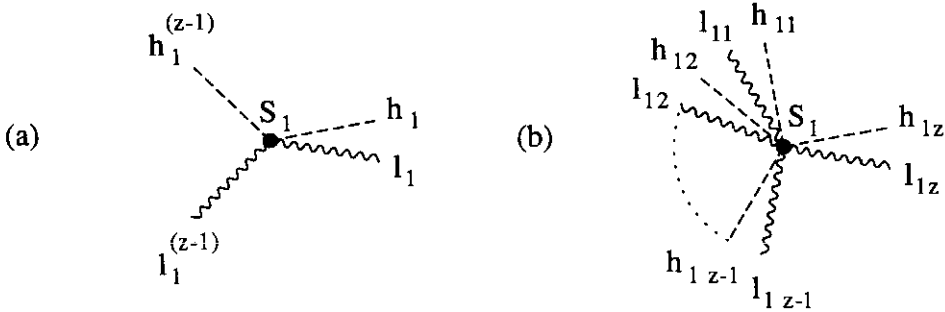
$$H = H \left( h_2^{(z-1)}, l_2^{(z-1)}, J, U \right) \\ = \frac{1}{\beta} \tanh^{-1} \left[ \frac{Z_1 \left( h_2^{(z-1)} + J, l_2^{(z-1)} + U \right) - Z_1 \left( h_2^{(z-1)} - J, l_2^{(z-1)} + U \right)}{Z_1 \left( h_2^{(z-1)} + J, l_2^{(z-1)} + U \right) + Z_1 \left( h_2^{(z-1)} - J, l_2^{(z-1)} + U \right)} \right] \tag{2.10}$$

$$L = L \left( h_2^{(z-1)}, l_2^{(z-1)}, J, U \right) \\ = \frac{1}{2\beta} \ln \left[ \frac{Z_1 \left( h_2^{(z-1)} + J, l_2^{(z-1)} + U \right) Z_1 \left( h_2^{(z-1)} - J, l_2^{(z-1)} + U \right)}{\left[ Z_1 \left( h_2^{(z-1)}, l_2^{(z-1)} \right) \right]^2} \right]. \tag{2.11}$$

For a general  $m$ -bond field  $h_i^{(m)}$  and  $m$ -bond potential  $l_i^{(m)}$  at site  $i$ , resulting from  $m$  neighbours  $1, \dots, m$ , we now make the following assumption

$$h_i^{(m)} = \sum_{k=1}^m h_{ik} \quad l_i^{(m)} = \sum_{k=1}^m l_{ik} \tag{2.12}$$

where  $h_{ik}$  ( $l_{ik}$ ) represents the effective field (potential) at site  $i$  from the  $k$ th neighbour through the interaction bond. The  $h_{ik}$ 's ( $l_{ik}$ 's) are called single-bond fields (potentials). See figure 3.



**Figure 3.** (a) Spin  $S_1$ , the single-bond field and potential  $h_1$  and  $l_1$ , and the  $(z - 1)$ -bond field and potential. (b) Spin  $S_1$  and all the single bond fields and potentials. Both figures are equivalent under the assumption (2.12) if  $h_1 \equiv h_{1z}$ ,  $l_1 \equiv l_{1z}$ .

The distribution function for the  $m$ -bond field  $h$  and  $m$ -bond potential  $l$  at the same site is denoted by  $g^{(m)}(h, l)$ , and the corresponding single-bond distribution function by  $g(h, l)$ .

$\langle \rangle_{12}$  denotes the thermal average with respect to  $\rho_{12}$ ,  $\langle \rangle_1$  denotes the thermal average with respect to  $\rho_1$ , and  $\langle \rangle_R$  denotes the random average.

Neglecting any correlations outside the pair-cluster, in the same spirit as our assumption (2.12), then gives

$$g^{(m)}(h, l) = \int \delta \left( h - \sum_{i=1}^m h_i \right) \delta \left( l - \sum_{i=1}^m l_i \right) \prod_{i=1}^m g(h_i, l_i) dh_i dl_i. \tag{2.13}$$

Furthermore, we introduce the consistency requirement  $\text{Tr}_2 \rho_{12} \propto \rho_1$ . This essentially means that a single-site expectation value in the pair approximation equals the same expectation value in the single-site approximation

$$\langle\langle F(S_1) \rangle\rangle_{12} = \langle\langle F(S_1) \rangle\rangle_1 \tag{2.14}$$

where  $F$  is an arbitrary function of  $S_1$ .

Using this consistency requirement and (2.13), IFT obtain

$$g^{(z-1)}(h, l) = \int \delta \left( h - \sum_{i=1}^{(z-1)} H(h_i, l_i, J_i, U_i) \right) \delta \left( l - \sum_{i=1}^{(z-1)} L(h_i, l_i, J_i, U_i) \right) \times \prod_{i=1}^{(z-1)} g^{(z-1)}(h_i, l_i) \mathcal{P}(J_i, U_i) dh_i dl_i dJ_i dU_i \tag{2.15}$$

and finally derive a set of coupled integral equations for the  $(z-1)$ -bond distribution function

$$g^{(z-1)}(h, l) = \frac{1}{(2\pi)^2} \int \exp[-i(ph + sl)] \cdot [G(p, h)]^{z-1} dp ds \tag{2.16}$$

$$G(p, s) = \int \exp[ipH(h, l, J, U) + isL(h, l, J, U)] \times \mathcal{P}(J, U) dJ dU g^{(z-1)}(h, l) dh dl. \tag{2.17}$$

The free energy per bond in the pair approximation is then obtained using stationarity arguments with respect to variations in the single-bond distribution function  $g(h, l)$  as

$$\varphi = \varphi_{12} - 2 \left( \frac{z-1}{z} \right) \varphi_1 \tag{2.18}$$

where

$$\varphi_{12} = -\frac{1}{\beta} \int \ln(\text{Tr} \rho_{12}) \mathcal{P}(J, U) dJ dU \prod_{k=1}^{z-1} g(h_{1k}, l_{1k}) dh_{1k} dl_{1k} \times \prod_{m=1}^{z-1} g(h_{2m}, l_{2m}) dh_{2m} dl_{2m} \tag{2.19}$$

$$\varphi_1 = -\frac{1}{\beta} \int \ln(\text{Tr} \rho_1) \prod_{k=1}^z g(h_{1k}, l_{1k}) dh_{1k} dl_{1k}. \tag{2.20}$$

The order parameters which classify the DMLG are given by

$$m \equiv \langle\langle S \rangle\rangle_R = \int \frac{\text{Tr}(S_1 \rho_{12})}{\text{Tr}(\rho_{12})} g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{2.21}$$

$$q \equiv \langle\langle S^2 \rangle\rangle_R = \int \left[ \frac{\text{Tr}(S_1 \rho_{12})}{\text{Tr}(\rho_{12})} \right]^2 g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{2.22}$$

$$\rho \equiv \langle\langle S^2 \rangle\rangle_R = \int \frac{\text{Tr}(S_1^2 \rho_{12})}{\text{Tr}(\rho_{12})} g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{2.23}$$

$$\lambda \equiv \langle\langle S^2 \rangle\rangle_R = \int \left[ \frac{\text{Tr}(S_1^2 \rho_{12})}{\text{Tr}(\rho_{12})} \right]^2 g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2. \tag{2.24}$$

For a more detailed exposé of the pair approximation, see IFT [14, 15].

In the limit of  $\mu \rightarrow \infty$  all lattice sites are occupied and the DMLG reduces to a standard disordered spin- $\frac{1}{2}$  system. In this case, and for a Gaussian distribution  $\mathcal{P}(J, U)$ , equations (2.21) and (2.22) become exactly the SK equations. In the case of no magnetic interaction,  $J_{ij} \rightarrow 0$ , the DMLG reduces to the DLG as a simple generalization of the Lee and Yang lattice gas. In the case of no potential interaction,  $U_{ij} \rightarrow 0$ , our model becomes what IFT call the simple disordered magnetic gas (SDMG). It is interesting to note that in this case and for a Gaussian distribution  $\mathcal{P}(J, U)$  the average effective potential  $\langle l \rangle$  does not necessarily vanish.

### 3. Equivalence of Cayley tree and pair approximation

Thompson, Inawashiro and Frankel (TIF) [11] have investigated the DMLG on a Cayley tree. They found that in the limit of infinite coordination number ( $z \rightarrow \infty$ ) the pair approximation corresponds to the DMLG on a Cayley tree if one considers local thermodynamic quantities ‘deep inside the Cayley tree’ rather than the corresponding global or bulk expressions. In this section we show that this equivalence can be established for arbitrary finite  $z$ .

The relation between the pair approximation and the Bethe lattice for the random bond Ising model has been discussed by Morita [18]. We make this discussion rigorous and extend it to the case of the DMLG. To this purpose, we first have to provide a precise definition of a local thermodynamic quantity of a system on a Cayley tree.

As one proceeds from the outermost shell towards the centre of a Cayley tree, the thermodynamic action of the ‘twigs’ or ‘branches’ can be entirely replaced by effective fields and potentials acting on the site from where the branches emerge. In this manner, an iteration scheme for the effective fields and potentials can be derived, given by (see TIF [11])

$$h_{i_1 \dots i_r} = \sum_{i_{r+1}=1}^{z-1} H(h_{i_1 \dots i_r i_{r+1}}, l_{i_1 \dots i_r i_{r+1}}, J_{i_1 \dots i_r i_{r+1}}, U_{i_1 \dots i_r i_{r+1}}) \tag{3.1}$$

$$l_{i_1 \dots i_r} = \sum_{i_{r+1}=1}^{z-1} L(h_{i_1 \dots i_r i_{r+1}}, l_{i_1 \dots i_r i_{r+1}}, J_{i_1 \dots i_r i_{r+1}}, U_{i_1 \dots i_r i_{r+1}}) \tag{3.2}$$

where  $(i_1 \dots i_r i_{r+1})$  denotes the  $i_{r+1}$ th neighbour of the lattice site  $(i_1 \dots i_r)$  as one walks towards the surface of the Cayley tree,  $i_1$  ranges from 1 to  $z$ , and all other  $i_k$ ’s range from 1 to  $(z - 1)$ .  $H(h, l, J, U)$  and  $L(h, l, J, U)$  are given by (2.10) and (2.11).  $h_{i_1 \dots i_r}$  ( $l_{i_1 \dots i_r}$ ) denotes the effective field (potential) acting from lattice site  $(i_1 \dots i_r)$  onto lattice site  $(i_1 \dots i_{r-1})$ . In particular,  $h_{i_1}$  ( $l_{i_1}$ ) are the  $z$  effective fields (potentials) acting on the central site  $S_0$ .  $J_{i_1 \dots i_{r+1}}$  ( $U_{i_1 \dots i_{r+1}}$ ) denotes the magnetic (potential) interaction between lattice sites  $(i_1 \dots i_r i_{r+1})$  and  $(i_1 \dots i_r)$ .

From the above recursion relations for the effective fields and potentials we can determine a recursion relation for the distribution function for these fields and potentials as we go from the outermost to the innermost shell, given by

$$\begin{aligned}
 &g_{i_1 \dots i_r}(h_{i_1 \dots i_r}, l_{i_1 \dots i_r}) \\
 &= \int \delta \left( h_{i_1 \dots i_r} - \sum_{i_{r+1}=1}^{z-1} H(h_{i_1 \dots i_{r+1}}, l_{i_1 \dots i_{r+1}}, J_{i_1 \dots i_{r+1}}, U_{i_1 \dots i_{r+1}}) \right) \\
 &\quad \times \delta \left( l_{i_1 \dots i_r} - \sum_{i_{r+1}=1}^{z-1} L(h_{i_1 \dots i_{r+1}}, l_{i_1 \dots i_{r+1}}, J_{i_1 \dots i_{r+1}}, U_{i_1 \dots i_{r+1}}) \right) \\
 &\quad \times \prod_{i_{r+1}=1}^{z-1} g_{i_1 \dots i_{r+1}}(h_{i_1 \dots i_{r+1}}, l_{i_1 \dots i_{r+1}}) \mathcal{P}(J_{i_1 \dots i_{r+1}}, U_{i_1 \dots i_{r+1}}) \\
 &\quad \times dh_{i_1 \dots i_{r+1}} dl_{i_1 \dots i_{r+1}} dJ_{i_1 \dots i_{r+1}} dU_{i_1 \dots i_{r+1}}. \tag{3.3}
 \end{aligned}$$

This is the most general form of the distribution functions.

For homogeneous boundary conditions, i.e. if all fields and potentials at the tips of the Cayley tree were equally distributed, we would have

$$g_{i_1 \dots i_r}(h, l) = g_r(h, l). \tag{3.4}$$

For independently distributed fields  $h$  and potentials  $l$  at the boundary of the Cayley tree we would further get

$$g_r(h, l) = g_r(h)g_r(l). \tag{3.5}$$

In order to be complete, the iteration schemes (3.1)–(3.3) obviously still require certain starting conditions which have to be specified at the surface of the Cayley tree.

### 3.1. Definition of local thermodynamic quantities for a system on a Cayley tree

*Definition A.* Thermodynamic quantities (per lattice site) which depend on single-site expectation values alone like the magnetization or the order parameters (2.21)–(2.24) are evaluated at the central site  $S_0$  of the Cayley tree. The effective fields and potentials acting on  $S_0$  are hereby taken as being distributed with a fixed point solution of the iteration scheme (3.3) for the distribution functions. This represents a formalisation of the intuition ‘deep inside an infinite Cayley tree’, and is then called a local thermodynamic quantity (per lattice site).

*Definition B.* Thermodynamic quantities (per lattice site) which depend on pair-site expectation values like the internal energy, the entropy, etc. can be derived from the free energy by differentiation. Thus, we only have to define a local free energy per lattice site,  $\psi_{\text{local}}$ . For consistency, we require that the just defined local magnetization  $m_{\text{local}}$  is recovered upon differentiation with respect to  $B$ , and that in the limiting case of  $\mathcal{P}(J, U)$  being a  $\delta$ -distribution we recover the non-random Bethe approximation. Thus, we define  $\psi_{\text{local}}$  by

$$-\frac{\partial}{\partial B} \psi_{\text{local}} = m_{\text{local}} \tag{3.6}$$

$$\lim_{B \rightarrow \infty} (\psi_{\text{local}}(B) + B) = E_0 \tag{3.7}$$



where  $E_0$  is the energy per lattice site in zero magnetic field of a corresponding regular lattice of coordination number  $z$  when all spins are  $+1$ . The interactions are hereby taken as the average magnetic and potential interactions,  $J_0 \equiv \langle J_{ij} \rangle$  and  $U_0 \equiv \langle U_{ij} \rangle$ .

We want to make two comments about this definition. First, equations (3.6) and (3.7) are equivalent to Thompson's integral definition of  $\psi_{\text{local}}$  [7]

$$\psi_{\text{local}} = \int_B^{\infty} (m_{\text{local}} - 1) dB + E_0 - B. \tag{3.8}$$

Second, the Bethe approximation for the non-random Ising model on a Cayley tree is only recovered if  $E_0 = Jz/2$ , which corresponds to our definition above, and not if  $E_0 = J$ , as would be the case if our lattice for evaluating  $E_0$  were the Cayley tree rather than the corresponding regular lattice of equal coordination number.

It is well known that the iteration scheme (3.3) has, in general, several fixed point solutions [13]. They correspond to different boundary conditions on the Bethe lattice which represent the initial values for the iteration. Thus, the local thermodynamic quantities defined in A and B will clearly depend on the specific boundary conditions on the Cayley tree.

### 3.2. Equivalence to the corresponding thermodynamic quantities in the pair approximation

*Proof A.* We first prove that single-site expectation values in the pair approximation and the corresponding local thermodynamic quantities on the Cayley tree are identical.

Assume that the boundary fields and potentials on the surface of the Cayley tree are randomly distributed according to a solution of the following integral equation

$$g^{(z-1)}(h, l) = \int \delta \left( h - \sum_{i=1}^{z-1} H(h_i, l_i, J_i, U_i) \right) \delta \left( l - \sum_{i=1}^{z-1} L(h_i, l_i, J_i, U_i) \right) \times \prod_{i=1}^{z-1} g^{(z-1)}(h_i, l_i) \mathcal{P}(J_i, U_i) dh_i dl_i dJ_i dU_i \tag{3.9}$$

which constitutes the distribution function for the  $(z - 1)$ -bond effective fields and potentials in the pair approximation, equation (2.15).

Inserting (3.9) into the recursion relation (3.3) now gives the result that any  $(z - 1)$ -bond distribution function for the pair approximation represents exactly a fixed point solution of equation (3.3), and vice versa.

According to our definition of a local thermodynamic quantity above, and taking into account equation (2.14) of the pair approximation for the DMLG, this means that local single-site expectation values on the Cayley tree and in the pair approximation are identical, for arbitrary  $z$ , i.e. in particular the order parameters

$$m = \langle \langle S_0 \rangle \rangle_R \quad q = \langle \langle S_0^2 \rangle \rangle_R \quad \rho = \langle \langle S_0^2 \rangle \rangle_R \quad \lambda = \langle \langle S_0^2 \rangle \rangle_R. \tag{3.10}$$

*Proof B.* It now remains to be shown that the free energies coincide as well.

The free energy per site for the DMLG in pair approximation is obtained from (2.18), (2.19) and (2.20) as

$$\psi = \varphi_1 + z \left( \frac{1}{2} \varphi_{12} - \varphi_1 \right). \quad (3.11)$$

This can easily be seen when considering the limit  $T \rightarrow 0$ . In order to show that (3.11) constitutes also the local free energy on a Cayley tree we have to prove that it satisfies (3.6) and (3.7). This is done in the appendix.

#### 4. A modified pair approximation

As described in the introduction, in their attempts to find a mean field theory for random spin systems which incorporates SRO both the pair-approximation club and authors who treated random systems on the Cayley tree derived a set of coupled integral equations for the distribution function of the effective fields which do not seem to be readily amenable to further analytical investigations. Although a very limited number of analytical results exist, like Katsura's solution of the integral equation for the  $\pm J$  model at  $T = 0$  [19] and some systematic  $1/z$  expansions [20], the majority of authors then simply continue their studies by either considering the limiting case of the infinite-range-interaction model ( $z \rightarrow \infty$ ), or by performing some numerical studies, or by discussing general properties of the system like the nature of the phase transitions.

Having shown that the method of the pair approximation is equivalent to a system on a Cayley tree when considering its local thermodynamic properties, we now present an alternative mean field theory which is based upon the pair approximation, and thus incorporates some kind of SRO, but which can be solved exactly for finite coordination number  $z$ .

##### 4.1. General formulation

The self-consistent approach of the pair approximation invariably leads to an intractable set of coupled integral equations for the distribution function of the effective fields. We, therefore, abandon the self-consistent approach now and suggest a mean field theory which treats the Kikuchi spin cluster of the pair approximation exactly but embeds it in zeroth-order effective fields and potentials from the infinite-range model. We let these zeroth-order fields and potentials dictate the general shape of the distribution function and then determine its specifying parameters self-consistently. The method will be formulated for a general random bond distribution  $\mathcal{P}(J, U)$  and will then be examined for the special case of  $\mathcal{P}(J, U)$  being Gaussian.

Let  $g^{(\infty)}(h, l)$  be the  $\infty$ -bond distribution function of the infinite-range model in pair approximation (see IFT [15]).  $g^{(\infty)}(h, l)$  constitutes the distribution function of the effective fields and potentials for a single-site expectation value. Hence,  $g^{(\infty)}(h, l)$  corresponds to the  $z$ -bond distribution function  $g^{(z)}(h, l)$  of a system with finite coordination number  $z$ . Embedding the Kikuchi pair cluster of the pair approximation for finite  $z$  in zeroth-order effective fields and potentials is then tantamount to making the following assumption

$$g^{(z)}(h, l) = g^{\infty}(h, l). \quad (4.1)$$

By defining the Fourier transforms

$$G_r(p, s) \equiv \int e^{i(ph+sl)} g^{(r)}(h, l) dh dl \tag{4.2}$$

and by using the theory of Fourier transforms, equations (2.12) and (2.13) yield

$$G_{z-1}(p, s) = [G_z(p, s)]^{(z-1)/z} . \tag{4.3}$$

From (4.1) and (4.2) we get

$$G_z(p, s) = \int e^{i(ph+sl)} g^\infty(h, l) dh dl. \tag{4.4}$$

Inserting (4.3) and (4.4) into equation (2.16) for the  $(z-1)$ -bond distribution function finally gives

$$g^{(z-1)}(h, l) = \frac{1}{(2\pi)^2} \int [G_z(p, s)]^{(z-1)/z} e^{-i(ph+sl)} dp ds. \tag{4.5}$$

Equations (4.5) and (4.4) are the general equations for determining the  $(z-1)$ -bond distribution function  $g^{(z-1)}(h, l)$  for our modified pair approximation.

$g^{(\infty)}(h, l)$  will still depend on the order parameters  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  from zeroth-order mean field theory. The actual order parameters for our model, so far, would be determined by inserting (4.5) and (4.3) into the defining order parameter equations (2.21)–(2.24) and would implicitly depend on  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  via exactly these equations.

However, we now make our approach ‘semi-self-consistent’. We replace  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  in the just mentioned equations by the order parameters  $m$ ,  $q$ ,  $\rho$  and  $\lambda$  which we want to determine and then determine them self-consistently from these equations. Following this procedure, we thus obtain

$$m \equiv \langle\langle S \rangle\rangle_R = \int \frac{\text{Tr}(S_1 \rho_{12})}{\text{Tr}(\rho_{12})} \tilde{g}^{(z-1)}(h_1, l_1) \tilde{g}^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{4.6}$$

$$q \equiv \langle\langle S^2 \rangle\rangle_R = \int \left[ \frac{\text{Tr}(S_1 \rho_{12})}{\text{Tr}(\rho_{12})} \right]^2 \tilde{g}^{(z-1)}(h_1, l_1) \tilde{g}^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{4.7}$$

$$\rho \equiv \langle\langle S^2 \rangle\rangle_R = \int \frac{\text{Tr}(S_1^2 \rho_{12})}{\text{Tr}(\rho_{12})} \tilde{g}^{(z-1)}(h_1, l_1) \tilde{g}^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{4.8}$$

$$\lambda \equiv \langle\langle S^2 \rangle\rangle_R = \int \left[ \frac{\text{Tr}(S_1^2 \rho_{12})}{\text{Tr}(\rho_{12})} \right]^2 \tilde{g}^{(z-1)}(h_1, l_1) \tilde{g}^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 \tag{4.9}$$

with  $\tilde{g}^{(z-1)}$  resulting from  $g^{(z-1)}$  if, in the determining equations (4.5) and (4.4), we replace  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  in  $g^{(\infty)}(h, l)$  by  $m$ ,  $q$ ,  $\rho$  and  $\lambda$ , respectively.

The order parameters  $m$ ,  $q$ ,  $\rho$ , and  $\lambda$  for our model are then determined self-consistently from equations (4.6)–(4.9). Once they are determined, they are inserted into  $g^{(\infty)}$ , hereby replacing  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  and thus yielding a modified function  $\tilde{g}^{(\infty)}(h, l)$ . Using this modified distribution  $\tilde{g}^{(\infty)}(h, l)$ , we can then explicitly determine the modified  $(z-1)$ -bond distribution function  $\tilde{g}^{(z-1)}(h, l)$  from (4.5) and (4.4).

All thermodynamic variables are then readily evaluated from the pair approximation equations (2.18)–(2.24) by using this modified  $(z-1)$ -bond distribution function.

4.2. Gaussian random bond distributions

We now examine the DMLG in our modified pair approximation for the special case of Gaussian random bonds.

For this purpose, we assume that the general probability distribution  $\mathcal{P}(J, U)$  is given by a product of two Gaussian forms,  $\mathcal{P}(J, U) = P(J) \cdot Q(U)$ , with

$$P(J) = \frac{1}{\sqrt{2\pi} \Delta J} \exp\left(-\frac{(J - J_0)^2}{2(\Delta J)^2}\right) \tag{4.10}$$

and

$$Q(U) = \frac{1}{\sqrt{2\pi} \Delta U} \exp\left(-\frac{(U - U_0)^2}{2(\Delta U)^2}\right) \tag{4.11}$$

where  $J_0$  ( $U_0$ ) denotes the average magnetic (potential) interaction and  $\Delta J$  ( $\Delta U$ ) is the width of the distribution for the magnetic (potential) interaction.

The case of infinitely-long-ranged interactions, or zeroth-order mean field theory, is derived from the pair approximation in the limit  $z \rightarrow \infty$ . For this purpose, however, the average interactions and widths of the above distributions have to be rescaled in order to yield finite thermodynamics per lattice site. We thus define

$$\begin{aligned} J_0 &\approx \lim_{z \rightarrow \infty} \frac{\tilde{J}_0}{z} & \Delta J &= \lim_{z \rightarrow \infty} \frac{\Delta \tilde{J}}{\sqrt{z}} \\ U_0 &= \lim_{z \rightarrow \infty} \frac{\tilde{U}_0}{z} & \Delta U &= \lim_{z \rightarrow \infty} \frac{\Delta \tilde{U}}{\sqrt{z}} \end{aligned} \tag{4.12}$$

with  $\tilde{J}_0$ ,  $\Delta \tilde{J}$ ,  $\tilde{U}_0$  and  $\Delta \tilde{U}$  being constants, and with  $\lim_{z \rightarrow \infty}$  being understood as the proper scaling as  $z \rightarrow \infty$  rather than the actual limit (which would be equal to 0).

Using these definitions and assumptions for Gaussian random bonds, IFT [15] have investigated the zeroth-order mean field limit  $z \rightarrow \infty$  and obtained the following  $\infty$ -bond distribution function for the effective fields and potentials

$$g^{(\infty)}(h, l) = g_M^{(\infty)}(h) \cdot g_L^{(\infty)}(l) \tag{4.13}$$

with

$$g_M^{(\infty)}(h) = \frac{1}{\sqrt{2\pi} \Delta h} \exp\left(-\frac{(h - h_0)^2}{2(\Delta h)^2}\right) \tag{4.14}$$

$$g_L^{(\infty)}(l) = \frac{1}{\sqrt{2\pi} \Delta l} \exp\left(-\frac{(l - l_0)^2}{2(\Delta l)^2}\right) \tag{4.15}$$

and

$$h_0 = m^{(\infty)} \tilde{J}_0 \tag{4.16}$$

$$l_0 = \rho^{(\infty)} \tilde{U}_0 + \frac{1}{2} \left(\rho^{(\infty)} - \lambda^{(\infty)}\right) \beta \left(\Delta \tilde{U}\right)^2 + \frac{1}{2} \left(\rho^{(\infty)} - q^{(\infty)}\right) \beta \left(\Delta \tilde{J}\right)^2 \tag{4.17}$$

$$\Delta h = \sqrt{q^{(\infty)}} \Delta \tilde{J} \tag{4.18}$$

$$\Delta l = \sqrt{\lambda^{(\infty)}} \Delta \tilde{U}. \tag{4.19}$$

Following the general procedure for our modified pair approximation, we first define the  $z$ -bond distribution function as

$$g^{(z)}(h, l) = g^{(\infty)}(h, l) \tag{4.20}$$

with the appropriate scaling

$$\begin{aligned} \tilde{J}_0 &= zJ_0 & \tilde{U}_0 &= zU_0 \\ \Delta \tilde{J} &= \sqrt{z} \Delta J & \Delta \tilde{U} &= \sqrt{z} \Delta U. \end{aligned} \tag{4.21}$$

Inserting  $g^{(\infty)}(h, l)$ , i.e. equations (4.13)–(4.19), into (4.4), gives

$$G_z(p, s) = \exp\left(-\frac{p^2(\Delta h)^2}{2} - \frac{s^2(\Delta l)^2}{2}\right) \exp(-ih_0p - il_0s) \tag{4.22}$$

and by using this result in the determining equation for the  $(z - 1)$ -bond distribution function, equation (4.5), we get

$$g^{(z-1)}(h, l) = \frac{1}{\sqrt{2\pi} \Delta h'} \exp\left(-\frac{(h - h'_0)^2}{2(\Delta h')^2}\right) \frac{1}{\sqrt{2\pi} \Delta l'} \exp\left(-\frac{(l - l'_0)^2}{2(\Delta l')^2}\right) \tag{4.23}$$

with

$$\begin{aligned} h'_0 &= \frac{z-1}{z} h_0 & \Delta h' &= \sqrt{\frac{z-1}{z}} \Delta h \\ l'_0 &= \frac{z-1}{z} l_0 & \Delta l' &= \sqrt{\frac{z-1}{z}} \Delta l. \end{aligned} \tag{4.24}$$

$h'_0$ ,  $\Delta h'$ ,  $l'_0$  and  $\Delta l'$  still depend on the order parameters  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  from the zeroth-order mean field approximation. Following the procedure for our modified pair approximation, we now replace  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  by  $m$ ,  $q$ ,  $\rho$ , and  $\lambda$  of our model and then determine them self-consistently from equations (4.6)–(4.9).

Replacing  $m^{(\infty)}$ ,  $q^{(\infty)}$ ,  $\rho^{(\infty)}$  and  $\lambda^{(\infty)}$  in (4.16)–(4.19) by  $m$ ,  $q$ ,  $\rho$ , and  $\lambda$ , respectively, and inserting this into equation (4.23) for the  $(z - 1)$ -bond distribution function gives the modified  $(z - 1)$ -bond distribution function  $\tilde{g}^{(z-1)}$  as

$$\tilde{g}^{(z-1)}(h, l) = \frac{1}{\sqrt{2\pi} \Delta \tilde{h}} \exp\left(-\frac{(h - \tilde{h}_0)^2}{2(\Delta \tilde{h})^2}\right) \frac{1}{\sqrt{2\pi} \Delta \tilde{l}} \exp\left(-\frac{(l - \tilde{l}_0)^2}{2(\Delta \tilde{l})^2}\right) \tag{4.25}$$

with

$$\tilde{h}_0 = (z - 1)mJ_0 \tag{4.26}$$

$$\tilde{l}_0 = (z - 1) \left[ \rho U_0 + \frac{1}{2}(\rho - \lambda)\beta(\Delta U)^2 + \frac{1}{2}(\rho - q)\beta(\Delta J)^2 \right] \tag{4.27}$$

$$\Delta \tilde{h} = \sqrt{z - 1} \sqrt{q} \Delta J \tag{4.28}$$

$$\Delta \tilde{l} = \sqrt{z - 1} \sqrt{\lambda} \Delta U. \tag{4.29}$$

The order parameters  $m$ ,  $q$ ,  $\rho$  and  $\lambda$  are hereby determined self-consistently from the set of integral equations (4.6)–(4.9), with  $\tilde{g}^{(z-1)}(h, l)$  being given by (4.25).

The free energy and other thermodynamic quantities are finally obtained by using this  $\tilde{g}^{(z-1)}(h, l)$  as the  $(z-1)$ -bond distribution function in equations (2.18)–(2.24) of the finite  $z$  pair approximation.

Equations (4.25)–(4.29) together with equations (4.6)–(4.9) represent our modified pair approximation for the case of Gaussian random bonds. If we now compare them with equations (4.13)–(4.19) and (2.21)–(2.24) of the pair approximation in the limit  $z \rightarrow \infty$ , the zeroth-order mean field limit, we discover that both sets of equations can be mapped into each other by a simple transformation. More specifically, our modified pair approximation for the DMLG with Gaussian random bonds corresponds exactly to treating the DMLG in the zeroth-order mean field approximation but with a different random bond distribution  $\mathcal{P}'(J, U)$  of smaller widths and means, given by

$$\begin{aligned} \tilde{J}'_0 &= \frac{z-1}{z} \tilde{J}_0 & \Delta \tilde{J}' &= \sqrt{\frac{z-1}{z}} \Delta \tilde{J} \\ \tilde{U}'_0 &= \frac{z-1}{z} \tilde{U}_0 & \Delta \tilde{U}' &= \sqrt{\frac{z-1}{z}} \Delta \tilde{U}. \end{aligned} \quad (4.30)$$

The modified pair approximation has been an exactly solvable attempt to include SRO into a mean field theory for the DMLG. At least for the case of Gaussian random bonds, equation (4.30) suggests that this inclusion of SRO thus simply amounts to a reduction in the randomness and strength of the interactions in the infinite-range model of zeroth-order mean field theory.

The SK equations are a special case of the DMLG in zeroth-order mean field approximation [15]. A numerical plot of the entropy from the SK solution shows that the entropy increases as randomness, i.e.  $\Delta \tilde{J}$ , decreases [10]. But this is exactly what the inclusion of SRO via our modified pair approximation generates. Since numerical studies of the entropy on a Bethe lattice [10] seem to suggest a positive entropy below the spin glass transition temperature, our modified pair approximation pushes mean-field theory in the right direction. However, as can be seen from equation (4.30), it still only represents a very marginal improvement indeed upon zeroth-order mean field theory, and certainly cannot resolve the negative entropy problem.

## 5. Summary and discussion

The disordered magnetic lattice gas (DMLG) as a unifying description of many simpler random spin models for amorphous solids and real fluids has been investigated in an attempt to devise a mean field theory which goes beyond the infinitely-long-ranged model by incorporating short-range order (SRO).

The usual procedure for incorporating SRO into a mean field theory for spin systems consists in treating them on a Cayley tree of finite coordination number  $z$ . In order to avoid peculiar Müller-Hartmann-Zittartz type solutions [21], and in order to recover the Bethe-Peierls-Weiss approximation in the non-random case and the infinite-range model in the limit  $z \rightarrow \infty$ , Thompson [6, 7] suggested that the proper procedure for treating a system on a Cayley tree is to consider its *local* thermodynamic properties rather than the corresponding global or bulk expressions.

The pair approximation together with the method of the distribution function represents an alternative approach to developing a mean field theory with SRO. This approach has been pursued by a series of different authors, see [14, 15, 17] and references therein.

We have shown rigorously in this paper that considering the local thermodynamic properties of the DMLG on a Cayley tree is equivalent to studying the DMLG in the pair approximation by the method of the distribution function, for arbitrary coordination number  $z$ . To this purpose we had to provide a precise definition of local thermodynamic quantities of a system on a Cayley tree. In contrast to Thompson, who determines local order parameters as the fixed points of an emerging iteration scheme for the order parameters, we consider the fixed-point solution of an iteration scheme for the distribution functions of the effective fields and potentials. The definition of local thermodynamic quantities is independent of boundary conditions. We could show that all local thermodynamic quantities for the DMLG on a Cayley tree coincide with those obtained by Inawashiro, Frankel and Thompson (IFT) [14] for the DMLG in the pair approximation, for arbitrary  $z$ .

Having established the equivalence of Cayley tree and pair approximation, we then presented a modified pair approximation as an exactly solvable attempt to incorporate SRO into a mean field theory for the DMLG. In this modified pair approximation the Kikuchi pair cluster is not embedded in self-consistent effective fields and potentials as in the pair approximation. Instead, the general shape of the distribution function for these fields and potentials is determined by the infinite-range model of zeroth-order mean field theory, while the specifying parameters of the distribution are then determined self-consistently.

After having formulated the method for general random bond distributions  $\mathcal{P}(J, U)$ , we examined the particular case of  $\mathcal{P}(J, U)$  being a Gaussian form. The resulting solution turns out to correspond exactly to an infinite-range model of the DMLG with a different Gaussian random bond distribution of smaller widths and means. Thus, the randomness and strength of the interactions in an infinite-range model of zeroth-order mean field theory are being reduced by the inclusion of SRO via our modified pair approximation. Although this appears to be physically reasonable in that it leads to an increase in the low temperature entropy, the modified pair approximation, at least in the case of Gaussian random bonds, turns out to constitute only a small improvement upon the infinite-range model, and clearly cannot solve the negative entropy problem.

Despite the rather modest outcome of our modified pair approximation, the techniques employed would lend themselves readily to the formulation of a 'zeroth-order theory' for correlation functions in random spin-1 systems, which, to the best of our knowledge, still constitutes relatively virgin ground. The idea behind this is to consider pair correlations on a Cayley tree by embedding a string-shaped spin cluster in zeroth-order effective fields and potentials. We expect that a richness of structure should arise in this case from the interplay between the phase criticality provided by the effective fields and potentials of the infinite-range model and the complexity of the one-dimensional random field Ising model which manifests itself already for the spin- $\frac{1}{2}$  model at  $T = 0$  [22]. Such an investigation is intended for a future publication.

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**Appendix**

In this appendix we are going to show that the free energy (3.11) satisfies equations (3.6) and (3.7) and thus constitutes the local free energy on a Cayley tree.

From (2.20), (2.2) and (2.12) we get

$$-\frac{\partial \varphi_1}{\partial B} = \langle \langle S_1 \rangle_1 \rangle_R = m_{\text{local}} \tag{A.1}$$

where we have used our definition of  $m_{\text{local}}$  and proof A from section 3.2 in the last equality.

From (2.19), (2.3) and (2.12) we get

$$-\frac{\partial}{\partial B} \frac{\varphi_{12}}{2} = \frac{1}{2} \left( \langle \langle S_1 \rangle_{12} \rangle_R + \langle \langle S_2 \rangle_{12} \rangle_R \right) \tag{A.2}$$

By using (2.14), our definition of  $m_{\text{local}}$  and noting proof A from section 3.2, we can write this as

$$-\frac{\partial}{\partial B} \frac{\varphi_{12}}{2} = \frac{1}{2} \left( \langle \langle S_1 \rangle_1 \rangle_R + \langle \langle S_2 \rangle_2 \rangle_R \right) = \langle \langle S_1 \rangle_1 \rangle_R = m_{\text{local}}. \tag{A.3}$$

Inserting (A.1) and (A.3) into (3.11) finally gives

$$-\frac{\partial}{\partial B} \psi = m_{\text{local}}. \tag{A.4}$$

This proves the first equality in the definition of the local free energy. We now have to show that  $\psi$  satisfies (3.7) as well.

From (2.20), (2.12) and (2.13) we have

$$\lim_{B \rightarrow \infty} [\varphi_1(B) + B] = \lim_{B \rightarrow \infty} \left[ -\frac{1}{\beta} \int \ln (1 + 2 \exp [\beta(\mu + l)] \cosh \beta(B + h)) g^{(z)}(h, l) dh dl + B \right]. \tag{A.5}$$

Taking the limit  $B \rightarrow \infty$  then yields

$$\lim_{B \rightarrow \infty} [\varphi_1(B) + B] = -\mu - \int (l + h) g^{(z)}(h, l) dh dl. \tag{A.6}$$



From (2.19), (2.12) and (2.13) we further have

$$\begin{aligned} & \lim_{B \rightarrow \infty} \left[ \frac{1}{2} \varphi_{12}(B) + B \right] \\ &= \lim_{B \rightarrow \infty} \left[ -\frac{1}{2\beta} \int \ln(\text{Tr} \rho_{12}) g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) \right. \\ & \quad \left. \times \mathcal{P}(J, U) dJ dU dh_1 dl_1 dh_2 dl_2 + B \right]. \end{aligned} \tag{A.7}$$

Inserting (2.8) and (2.6) gives

$$\begin{aligned} & \lim_{B \rightarrow \infty} \left[ \frac{1}{2} \varphi_{12}(B) + B \right] \\ &= \lim_{B \rightarrow \infty} \left[ -\frac{1}{2\beta} \int \ln \left( 1 + 2 \exp[\beta(\mu + l_1)] \cosh \beta(B + h_1) \right. \right. \\ & \quad \left. \left. + 2 \exp[\beta(\mu + l_2)] \cosh \beta(B + h_2) \right. \right. \\ & \quad \left. \left. + 4 \exp[\beta(2\mu + l_1 + l_2 + U)] \right. \right. \\ & \quad \left. \left. \times \left( \cosh \beta J \cosh \beta(B + h_1) \cosh \beta(B + h_2) \right. \right. \right. \\ & \quad \left. \left. \left. + \sinh \beta J \sinh \beta(B + h_1) \sinh \beta(B + h_2) \right) \right) \right. \\ & \quad \left. \times \mathcal{P}(J, U) dJ dU g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) dh_1 dl_1 dh_2 dl_2 + B \right] \\ &= -\frac{1}{2\beta} \int \ln \left( \exp[\beta(2\mu + l_1 + l_2 + U + J + h_1 + h_2)] \right) \\ & \quad \times g^{(z-1)}(h_1, l_1) g^{(z-1)}(h_2, l_2) \mathcal{P}(J, U) dJ dU dh_1 dl_1 dh_2 dl_2 \end{aligned} \tag{A.8}$$

which upon integration becomes

$$\lim_{B \rightarrow \infty} \left[ \frac{1}{2} \varphi_{12} + B \right] = -\mu - J_0 \frac{1}{2} - U_0 \frac{1}{2} - \int (h + l) g^{(z-1)}(h, l) dh dl. \tag{A.9}$$

From the Hamiltonian (2.1) and our definition of  $E_0$  following equation (3.7) we further have

$$E_0 = -J_0 \frac{z}{2} - U_0 \frac{z}{2} - \mu. \tag{A.10}$$

Inserting (A.6), (A.9) and (A.10) into equation (3.11) for the free energy in the pair approximation  $\psi$ , finally gives

$$\lim_{B \rightarrow \infty} (\psi + B) = E_0 \tag{A.11}$$

where we have used equations (2.12) and (2.13) to establish that  $(z-1)\langle l^{(z)} \rangle = z\langle l^{(z-1)} \rangle$  and  $(z-1)\langle h^{(z)} \rangle = z\langle h^{(z-1)} \rangle$ .

Combining equations (A.4) and (A.11) completes the proof that the free energy for the DMLG in the pair approximation is identical to the local free energy of the DMLG on a Cayley tree, for arbitrary coordination number  $z$ .

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